So far . . .

Free Lagrangians:

$$\mathcal{L}_0 = \tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + \tfrac{1}{2}\left(\tfrac{mc}{\hbar}\right)^2\phi^2 \qquad \text{K-G}$$

$$\mathcal{L}_{1/2} = \hbar c\,\bar{\psi}\gamma^\mu\partial_\mu\psi + mc^2\bar{\psi}\psi \qquad \text{Dirac}$$

$$\mathcal{L}_1 = \tfrac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{8\pi}\left(\tfrac{mc}{\hbar}\right)^2 A_\nu A^\nu \qquad \text{Proca}$$

All imply $\frac{E^2}{c^2} - p^2 = m^2c^2 = -p_\mu p^\mu$

We start with spin-$\frac{1}{2}$ matter $\mathcal{L}_{1/2}$ that is invariant under some global transformation Lie group G w/ generators λ_i, i.e. $\mathcal{L}(\psi) \to \mathcal{L}(e^{-ig\vec{\lambda}\cdot\vec{\phi}}\psi) = \mathcal{L}(\psi)$ where g is the coupling, λ is the "vector" of generators, and $\vec{\phi}$ is the vector of parameters.

Then we introduce interactions by promoting global $\vec{\phi}$ to local $\vec{\phi}(x^\mu)$.

To do so we must redefine the derivative to a covariant form:

$$\partial_\mu \to D_\mu = \partial_\mu + ig\lambda\cdot A_\mu \;\Rightarrow\; D_\mu\psi \to D_\mu'\psi' = e^{-ig\vec{\lambda}\cdot\vec{\phi}}D_\mu\psi$$

where the covariance requires $\underbrace{\lambda\cdot A_\mu \to \lambda\cdot A_\mu' = e^{-ig\vec{\lambda}\cdot\vec{\phi}}\lambda\cdot A_\mu e^{ig\vec{\lambda}\cdot\vec{\phi}} + \tfrac{i}{g}\partial_\mu\left(e^{-ig\vec{\lambda}\cdot\vec{\phi}}\right)e^{ig\vec{\lambda}\cdot\vec{\phi}}}_{A_\mu \text{ transforms in the "adjoint" representation of } G.}$

To allow the new gauge fields A_μ^i to propagate we add in the Proca Lagrangian w/ $m_{A_\mu^i}=0$ for invariance.

$$\tfrac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} \quad \text{w/} \quad F_{\mu\nu} = -\tfrac{i}{g}[D_\mu, D_\nu]$$

E&M: $U(1)$ on complex ψ

QCD: $SU(3)$ on $\psi = \begin{pmatrix}\psi_r\\ \psi_b\\ \psi_g\end{pmatrix}$

These two results do <u>not</u> agree w/ experiment

EW: $\underbrace{SU(2)_L}_{\text{acts on } \chi_L} \times \underbrace{U(1)_Y}_{\text{acts on } \chi_L \text{ and } e_R}$ on $\chi_L = \begin{pmatrix}\nu_e\\ e\end{pmatrix}_L$, etc. $\quad e_R$, etc.

Due to the fact that $SU(2)_L$ acts only on the left, all matter must have $m=0$ since the mass terms $mc^2\bar{\psi}_R\psi_L + mc^2\bar{\psi}_L\psi_R$ are not invariant.

This one has an interesting resolution. To avoid unnecessary complications let's consider a simpler version based on $U(1)$.

(Instead of $SU(2)_L \times U(1)_Y$ which is the real story but much nastier!)

Instead of starting w/ a spinor field, we will start w/ a complex scalar for reasons which will eventually be obvious.

Our starting point is a kinetic term for the scalar plus two self-interaction terms:

$$\mathcal{L} = \underbrace{\tfrac{1}{2}(\partial_\mu\phi)^*(\partial^\mu\phi)}_{\text{usual spin-0 kinetic}} \overbrace{\underbrace{-\tfrac{1}{2}\mu^2\phi^*\phi}_{\text{``wrong sign'' mass term}} + \underbrace{\tfrac{1}{4}\lambda^2(\phi^*\phi)^2}_{\text{quartic self-interaction}}}^{\text{Let's just call this } U(\phi,\phi^*)} \qquad \text{where } \phi = \phi_1 + i\phi_2$$

This may seem interesting since $m^2<0 \Rightarrow \frac{E^2}{c^2} - p^2 < 0 \Rightarrow \underbrace{\gamma^2 m^2 v^2}_{p^2} > \underbrace{\gamma^2 m^2 c^2}_{\frac{E^2}{c^2} = \frac{(\gamma m c^2)^2}{c^2}} \Rightarrow v^2 > c^2 \Rightarrow$ tachyonic ?!
BUT... we will learn how to interpret this in a more sensible way in field theory.

Since ϕ is complex, we notice that this $\mathcal{L}$ has a global $U(1)$ symmetry, so we can play the familiar gauging game:

1. $\phi \to e^{i\theta}\phi \Rightarrow \phi^* \to e^{-i\theta}\phi^* \Rightarrow \mathcal{L} \to \mathcal{L}$

2. $\phi \to e^{i\theta(x)}\phi(x) \Rightarrow \partial_\mu \to D_\mu \equiv \partial_\mu + i\frac{q}{\hbar c}A_\mu$ w/ $A_\mu \to A_\mu' = A_\mu - \frac{\hbar c}{q}\partial_\mu\theta$

3. Add $\mathcal{L}_{\text{proca}}$ w/ $m=0$ for A_μ to obtain:

$$\mathcal{L} = \tfrac{1}{2}\left[(\partial_\mu - \tfrac{iq}{\hbar c}A_\mu)\phi^*\right]\left[(\partial^\mu + \tfrac{iq}{\hbar c}A^\mu)\phi\right] - \tfrac{1}{2}\mu^2\phi^*\phi + \tfrac{1}{4}\lambda^2(\phi^*\phi)^2 + \tfrac{1}{16\pi}\underbrace{F^{\mu\nu}}_{\partial^\mu A^\nu - \partial^\nu A^\mu}F_{\mu\nu}$$

So far we have a gauge theory w/ a massless gauge field.

However when we "do" particle physics what we are really studying are small fluctuations in the fields (one for each type of particle). But what is the larger "background" configuration of the fields (above which we study fluctuations)? We may assume it is zero, but is that consistent and are there other options?

We get the background field configurations by solving the classical e.o.m. from $\mathcal{L}$, then treat the small fluctuations quantum mechanically.

This is what underlies the spirit of Feynman diagrams, i.e. perturbative QFT. Backgrounds are decidedly non-perturbative.

Okay so let's see this in action:

If $\mathcal{L} = T + U(\phi,\phi^*)$ then the simplest solutions for backgrounds come from setting: $\partial\phi \to 0 \Rightarrow T = 0$

then: $\frac{\partial U}{\partial \phi^*} = 0$ solves e.o.m.

$$\mathcal{L}(\phi,\phi^*,A_\mu) = \underbrace{\tfrac{1}{2}\left[\left(\partial_\mu - \tfrac{iq}{\hbar c}A_\mu\right)\phi^*\right]\left[\left(\partial^\mu + \tfrac{iq}{\hbar c}A^\mu\right)\phi\right]}_{T_\phi} \underbrace{- \tfrac{1}{2}\mu^2\phi^*\phi + \tfrac{1}{4}\lambda^2(\phi^*\phi)^2}_{U(\phi,\phi^*)} + \underbrace{\tfrac{1}{16\pi}F^{\mu\nu}F_{\mu\nu}}_{T_{A^\nu}}$$

$\Downarrow$

$$\frac{\partial U}{\partial \phi^*} = -\frac{1}{2}\mu^2\phi + \frac{1}{2}\lambda^2\phi^*\phi\,\phi = -\frac{1}{2}\mu^2\phi + \frac{1}{2}\lambda^2|\phi|^2\phi$$

i) One solution is $A_\mu = 0,\ \phi = 0$

Using this solution and studying $\phi(x) = 0 + \delta\phi(x)$
$A_\mu(x) = 0 + \delta A_\mu(x)$

We have: $\mathcal{L}(\delta\phi,\delta\phi^*,\delta A_\mu) = \frac{1}{2}\left[\left(\partial_\mu - \frac{iq}{\hbar c}\delta A_\mu\right)\delta\phi^*\right]\left[\left(\partial^\mu + \frac{iq}{\hbar c}\delta A^\mu\right)\delta\phi\right] - \frac{1}{2}\mu^2\delta\phi^*\delta\phi + \frac{1}{4}\lambda^2(\delta\phi^*\delta\phi) + \frac{1}{16\pi}\underbrace{F^{\mu\nu}F_{\mu\nu}}_{\text{from } \delta A_\mu}$

This looks exactly like what we had before, just w/ $\phi \to \delta\phi$, $\phi^* \to \delta\phi^*$, $A_\mu \to \delta A_\mu$.

ii) But another is $A_\mu = 0,\ \phi = \phi_0$ where ϕ_0 satisfies $|\phi_0|^2 = \frac{\mu^2}{\lambda^2} = \phi_{10}^2 + \phi_{20}^2$ (since $-\frac{1}{2}\mu^2\phi_0 + \frac{1}{2}\lambda^2|\phi_0|^2\phi_0 = 0$)

Using the specific choice $\phi_{10} = \frac{\mu}{\lambda}$, $\phi_{20} = 0$, $A_\mu = 0$ and studying $\phi_1(x) = \frac{\mu}{\lambda} + \delta\phi_1(x) \equiv \frac{\mu}{\lambda} + \eta(x)$, $\phi_2(x) = 0 + \delta\phi_2(x) \equiv \beta(x)$, $A_\mu(x) = 0 + \delta A_\mu(x) \equiv A_\mu(x)$ we find:

$$\mathcal{L} = \left[\tfrac{1}{2}(\partial_\mu\eta)(\partial^\mu\eta) + \mu^2\eta^2\right] + \left[\tfrac{1}{2}(\partial_\mu\beta)(\partial^\mu\beta)\right] + \left[+\tfrac{1}{16\pi}F^{\mu\nu}F_{\mu\nu} + \tfrac{1}{2}\left(\tfrac{q}{\hbar c}\tfrac{\mu}{\lambda}\right)^2 A_\mu A^\mu\right]$$
$$+ \left\{\tfrac{q}{\hbar c}\left[\eta(\partial_\mu\beta) - \beta(\partial_\mu\eta)\right]A^\mu + \tfrac{\mu}{\lambda}\left(\tfrac{q}{\hbar c}\right)^2\eta(A_\mu A^\mu) + \tfrac{1}{2}\left(\tfrac{q}{\hbar c}\right)^2(\beta^2+\eta^2)A_\mu A^\mu\right.$$
$$\left.+ \lambda\mu(\eta^3 + \eta\beta^2) + \tfrac{1}{4}\lambda^2(\eta^4 + 2\eta^2\beta^2 + \beta^4)\right\} + \left(\tfrac{\mu}{\lambda}\tfrac{q}{\hbar c}\right)(\partial_\mu\beta)A^\mu - \left(\tfrac{\mu^2}{2\lambda}\right)^2$$

What does this describe?

- A massive real scalar field η w/ $\mu^2 = \frac{1}{2}\left(\frac{m_\eta c}{\hbar}\right)^2 \Rightarrow m_\eta = \frac{\sqrt{2}\hbar}{c}\mu$
- A massive gauge field A_μ w/ $\frac{1}{2}\left(\frac{q}{\hbar c}\frac{\mu}{\lambda}\right)^2 = \frac{1}{8\pi}\left(\frac{m_A c}{\hbar}\right)^2 \Rightarrow m_A = 2\sqrt{\pi}\left(\frac{q\mu}{\lambda c^2}\right)$ Boom!!
- A massless scalar β
- All are interacting w/ each other in weird ways.

What you have just seen is the simplest example of the "Higgs mechanism."